THE PARABOLIC INFINITE-LAPLACE EQUATION IN CARNOT GROUPS

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Abstract. By employing a Carnot parabolic maximum principle, we show existence-uniqueness of viscosity solutions to a class of equations modeled on the parabolic infinite Laplace equation in Carnot groups. We show stability of solutions within the class and examine the limit as t goes to infinity.

1. Motivation

In Carnot groups, the following theorem has been established.

Theorem 1.1. [3, 16, 5] *Let Ω be a bounded domain in a Carnot group and let $v : \partial\Omega \to \mathbb{R}$ be a continuous function. Then the Dirichlet problem*

$$\begin{cases} \Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = v & \text{on} \quad \partial\Omega \end{cases}$$

has a unique viscosity solution u_∞.

Our goal is to prove a parabolic version of Theorem 1.1 for a class of equations (defined in the next section), namely

Conjecture 1.2. *Let Ω be a bounded domain in a Carnot group and let $T > 0$. Let $\psi \in C(\overline{\Omega})$ and let $g \in C(\Omega \times [0,T))$ Then the Cauchy-Dirichlet problem*

$$\begin{cases} u_t - \Delta_\infty^h u = 0 & \text{in } \Omega \times (0,T), \\ u(x,0) = \psi(x) & \text{on } \overline{\Omega} \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{1.1}$$

has a unique viscosity solution u.

In Sections 2 and 3, we review key properties of Carnot groups and parabolic viscosity solutions. In Section 4, we prove uniqueness and Section 5 covers existence.

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2. Calculus on Carnot Groups

We begin by denoting an arbitrary Carnot group in $\mathbb{R}^N$ by G and its corresponding Lie Algebra by g. Recall that g is nilpotent and stratified, resulting in the decomposition

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

for appropriate vector spaces that satisfy the Lie bracket relation $[V_1, V_j] = V_{1+j}$. The Lie Algebra g is associated with the group G via the exponential map $\exp : g \to G$. Since this map is a diffeomorphism, we can choose a basis for g so that it is the identity map. Denote this basis by

$$X_1, X_2, \ldots, X_{n_1}, Y_1, Y_2, \ldots, Y_{n_2}, Z_1, Z_2, \ldots, Z_{n_3}$$

so that

$$\begin{aligned} V_1 &= \operatorname{span}\{X_1, X_2, \ldots, X_{n_1}\} \\ V_2 &= \operatorname{span}\{Y_1, Y_2, \ldots, Y_{n_2}\} \\ V_3 \oplus V_4 \oplus \cdots \oplus V_l &= \operatorname{span}\{Z_1, Z_2, \ldots, Z_{n_3}\}. \end{aligned}$$

We endow g with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. Clearly, the Riemannian dimension of g (and so G) is $N = n_1 + n_2 + n_3$. However, we will also consider the homogeneous dimension of G, denoted $\mathcal{Q}$, which is given by

$$\mathcal{Q} = \sum_{i=1}^{l} i \cdot \dim V_i.$$

Before proceeding with the calculus, we recall the group and metric space properties. Since the exponential map is the identity, the group law is the Campbell-Hausdorff formula (see, for example, [7]). For our purposes, this formula is given by

$$p \cdot q = p + q + \frac{1}{2}[p, q] + R(p, q) \tag{2.1}$$

where $R(p, q)$ are terms of order 3 or higher. The identity element of G will be denoted by 0 and called the origin. There is also a natural metric on G, which is the Carnot-Carathéodory distance, defined for the points p and q as follows:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t) \in V_1$. By Chow's theorem (see, for example, [2]) any two points can be connected by such a curve, which means $d_C(p, q)$ is an honest metric. Define a Carnot-Carathéodory ball of radius r centered at a point p_0 by

$$B(p_0, r) = \{p \in G : d_C(p, p_0) < r\}.$$

In addition to the Carnot-Carathéodory metric, there is a smooth (off the origin) gauge. This gauge is defined for a point $p = (\zeta_1, \zeta_2, \ldots, \zeta_l)$ with $\zeta_i \in V_i$ by

$$\mathcal{N}(p) = \left(\sum_{i=1}^{l} \|\zeta_i\|^{\frac{2l!}{i}} \right)^{\frac{1}{2l!}} \tag{2.2}$$

and it induces a metric $d_{\mathcal{N}}$ that is bi-Lipschitz equivalent to the Carnot-Carathéodory metric and is given by

$$d_{\mathcal{N}}(p, q) = \mathcal{N}(p^{-1} \cdot q).$$

We define a gauge ball of radius r centered at a point p_0 by

$$B_{\mathcal{N}}(p_0, r) = \{p \in G : d_{\mathcal{N}}(p, p_0) < r\}.$$

In this environment, a smooth function $u : G \to \mathbb{R}$ has the horizontal derivative given by

$$\nabla_0 u = (X_1 u, X_2 u, \ldots, X_{n_1} u)$$

and the symmetrized horizontal second derivative matrix, denoted by $(D^2u)^\star$, with entries

$$((D^2u)^\star)_{ij} = \frac{1}{2}(X_i X_j u + X_j X_i u)$$

for $i, j = 1, 2, \ldots, n_1$. We also consider the semi-horizontal derivative given by

$$\nabla_1 u = (X_1 u, X_2 u, \ldots, X_{n_1} u, Y_1 u, Y_2 u, \ldots, Y_{n_2} u).$$

Using the above derivatives, we define the h-homogeneous infinite Laplace operator for $h \geq 1$ by

$$\Delta_\infty^h f = \|\nabla_0 f\|^{h-3} \sum_{i,j=1}^{n_1} X_i f X_j f X_i X_j f = \|\nabla_0 f\|^{h-3} \langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \rangle.$$

Given $T > 0$ and a function $u : G \times [0, T] \to \mathbb{R}$, we may define the analogous subparabolic infinite Laplace operator by

$$u_t - \Delta_\infty^h u$$

and we consider the corresponding equation

$$u_t - \Delta_\infty^h u = 0. \tag{2.3}$$

We note that when $h \geq 3$, this operator is continuous. When $h = 3$, we have the subparabolic infinite Laplace equation analogous to the infinite Laplace operator in [5]. The Euclidean analog for $h = 1$ has been explored in [14] and the Euclidean analog for $1 < h < 3$ in [15].

We recall that for any open set $\mathcal{O} \subset G$, the function f is in the horizontal Sobolev space $W^{1,\mathtt{p}}(\mathcal{O})$ if f and $X_i f$ are in $L^{\mathtt{p}}(\mathcal{O})$ for $i = 1, 2, \ldots, n_1$. Replacing $L^{\mathtt{p}}(\mathcal{O})$ by $L^{\mathtt{p}}_{loc}(\mathcal{O})$, the space $W^{1,\mathtt{p}}_{loc}(\mathcal{O})$ is defined similarly. The space $W^{1,\mathtt{p}}_0(\mathcal{O})$ is the closure in $W^{1,\mathtt{p}}(\mathcal{O})$ of smooth functions with compact support. In addition, we recall a function $u : G \to \mathbb{R}$ is $\mathcal{C}^2_{\text{sub}}$ if $\nabla_1 u$ and $X_i X_j u$ are continuous for all $i, j = 1, 2, \ldots n_1$. Note that $\mathcal{C}^2_{\text{sub}}$ is not equivalent to (Euclidean) C^2. For spaces involving time, the space $C(t_1, t_2; X)$ consists

of all continuous functions $u : [t_1, t_2] \to X$ with $\max_{t_1 \le t \le t_2} \|u(\cdot, t)\|_X < \infty$. A similar definition holds for $L^{\mathfrak{p}}(t_1, t_2; X)$.

Given an open box $\mathcal{O} = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)$, we define the parabolic space $\mathcal{O}_{t_1,t_2}$ to be $\mathcal{O} \times [t_1, t_2]$. Its parabolic boundary is given by $\partial_{\text{par}}\mathcal{O}_{t_1,t_2} = (\overline{\mathcal{O}} \times \{t_1\}) \cup (\partial\mathcal{O} \times (t_1, t_2])$.

Finally, recall that if G is a Carnot group with homogeneous dimension $\mathcal{Q}$, then $G \times \mathbb{R}$ is again a Carnot group of homogeneous dimension $\mathcal{Q} + 1$ where we have added an extra vector field $\frac{\partial}{\partial t}$ to the first layer of the grading. This allows us to give meaning to notations such as $W^{1,2}(\mathcal{O}_{t_1,t_2})$ and $\mathcal{C}^2_{\text{sub}}(\mathcal{O}_{t_1,t_2})$ where we consider $\nabla_0 u$ to be $\big(X_1 u, X_2 u, \ldots, X_{n_1} u, \frac{\partial u}{\partial t}\big)$.

3. Parabolic Jets and Viscosity Solutions

3.1. **Parabolic Jets.** In this subsection, we recall the definitions of the parabolic jets, as given in [6], but included here for completeness. We define the parabolic superjet of $u(p, t)$ at the point $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$, denoted $P^{2,+}u(p_0, t_0)$, by using triples $(a, \eta, X) \in \mathbb{R} \times V_1 \oplus V_2 \times S^{n_1}$ so that $(a, \eta, X) \in P^{2,+}u(p_0, t_0)$ if

$$\begin{aligned} u(p,t) \quad &\le \quad u(p_0,t_0) + a(t-t_0) + \langle \eta, \widehat{p_0^{-1}\cdot p}\rangle + \frac{1}{2}\langle X\overline{p_0^{-1}\cdot p}, \overline{p_0^{-1}\cdot p}\rangle \\ &\quad +o(|t-t_0| + |p_0^{-1}\cdot p|^2) \ \text{ as } \ (p,t) \to (p_0,t_0). \end{aligned}$$

We recall that S^k is the set of $k \times k$ symmetric matrices and $n_i = \dim V_i$. We define $\overline{p_0^{-1}\cdot p}$ as the first n_1 coordinates of $p_0^{-1}\cdot p$ and $\widehat{p_0^{-1}\cdot p}$ as the first $n_1 + n_2$ coordinates of $p_0^{-1}\cdot p$. This definition is an extension of the superjet definition for subparabolic equations in the Heisenberg group [4]. We define the subjet $P^{2,-}u(p_0, t_0)$ by

$$P^{2,-}u(p_0,t_0) = -P^{2,+}(-u)(p_0,t_0).$$

We define the set theoretic closure of the superjet, denoted $\overline{P}^{2,+}u(p_0, t_0)$, by requiring $(a, \eta, X) \in \overline{P}^{2,+}u(p_0, t_0)$ exactly when there is a sequence $(a_n, p_n, t_n, u(p_n, t_n), \eta_n, X_n) \to (a, p_0, t_0, u(p_0, t_0), \eta, X)$ with the triple $(a_n, \eta_n, X_n) \in P^{2,+}u(p_n, t_n)$. A similar definition holds for the closure of the subjet.

We may also define jets using appropriate test functions. Given a function $u : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$ we consider the set $\mathcal{A}u(p_0, t_0)$ given by

$$\mathcal{A}u(p_0,t_0) = \{\phi \in \mathcal{C}^2_{\text{sub}}(\mathcal{O}_{t_1,t_2}) : u(p,t) - \phi(p,t) \le u(p_0,t_0) - \phi(p_0,t_0) = 0 \ \forall (p,t) \in \mathcal{O}_{t_1,t_2}\}.$$

consisting of all test functions that touch u from above at (p_0, t_0). We define the set of all test functions that touch from below, denoted $\mathcal{B}u(p_0, t_0)$, similarly.

The following lemma relates the test functions to jets. The proof is identical to Lemma 3.1 in [4], but uses the (smooth) gauge $\mathcal{N}(p)$ instead of Euclidean distance.

Lemma 3.1.

$$P^{2,+}u(p_0,t_0) = \{(\phi_t(p_0,t_0), \nabla\phi(p_0,t_0), (D^2\phi(p_0,t_0))^\star) : \phi \in \mathcal{A}u(p_0,t_0)\}.$$

3.2. **Jet Twisting.** We recall that the set $V_1 = \text{span}\{X_1, X_2, \dots, X_{n_1}\}$ and notationally, we will always denote n_1 by n. The vectors X_i at the point $p \in G$ can be written as

$$X_i(p) = \sum_{j=1}^{N} a_{ij}(p) \frac{\partial}{\partial x_j}$$

forming the $n \times N$ matrix $\mathbb{A}$ with smooth entries $\mathbb{A}_{ij} = a_{ij}(p)$. By linear independence of the X_i, $\mathbb{A}$ has rank n. Similarly,

$$Y_i(p) = \sum_{j=1}^{N} b_{ij}(p) \frac{\partial}{\partial x_j}$$

forming the $n_2 \times N$ matrix $\mathbb{B}$ with smooth entries $\mathbb{B}_{ij} = b_{ij}$. The matrix $\mathbb{B}$ has rank n_2. The following lemma differs from [5, Corollary 3.2] only in that there is now a parabolic term. This term however, does not need to be twisted. The proof is then identical, as only the space terms need twisting.

Lemma 3.2. *Let $(a, \eta, X) \in \overline{P}^{2,+}_{\text{eucl}} u(p,t)$. (Recall that $(\eta, X) \in \mathbb{R}^N \times S^N$.) Then*

$$(a, \mathbb{A} \cdot \eta \oplus \mathbb{B} \cdot \eta,\ \mathbb{A} X \mathbb{A}^T + \mathbb{M}) \in \overline{P}^{2,+} u(p,t).$$

Here the entries of the (symmetric) matrix $\mathbb{M}$ are given by

$$\mathbb{M}_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{N} \sum_{l=1}^{N} \left(a_{il}(p) \frac{\partial}{\partial x_l} a_{jk}(p) + a_{jl}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \right) \eta_k & i \neq j, \\[2ex] \displaystyle\sum_{k=1}^{N} \sum_{l=1}^{N} a_{il}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \eta_k & i = j. \end{cases}$$

3.3. **Viscosity Solutions.** We consider parabolic equations of the form

$$u_t + F(t, p, u, \nabla_1 u, (D^2 u)^\star) = 0 \tag{3.1}$$

for continuous and proper $F : [0,T] \times G \times \mathbb{R} \times g \times S^n \to \mathbb{R}$. [8] We recall that S^n is the set of $n \times n$ symmetric matrices (where $\dim V_1 = n$) and the derivatives $\nabla_1 u$ and $(D^2 u)^\star$ are taken in the space variable p. We then use the jets to define subsolutions and supersolutions to Equation (3.1) in the usual way.

Definition 1. Let $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ be as above. The upper semicontinuous function u is a *parabolic viscosity subsolution* in $\mathcal{O}_{t_1,t_2}$ if for all $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ we have $(a, \eta, X) \in \overline{P}^{2,+} u(p_0, t_0)$ produces

$$a + F(t_0, p_0, u(p_0, t_0), \eta, X) \leq 0.$$

A lower semicontinuous function u is a *parabolic viscosity supersolution* in $\mathcal{O}_{t_1,t_2}$ if for all $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ we have $(b, \nu, Y) \in \overline{P}^{2,-} u(p_0, t_0)$ produces

$$b + F(t_0, p_0, u(p_0, t_0), \nu, Y) \geq 0.$$

A continuous function u is a *parabolic viscosity solution* in $\mathcal{O}_{t_1,t_2}$ if it is both a parabolic viscosity subsolution and parabolic viscosity supersolution.

Remark 3.3. *In the special case when* $F(t,p,u,\nabla_1 u,(D^2u)^\star) = F^h_\infty(\nabla_0 u,(D^2u)^\star) = -\Delta^h_\infty u$*, for* $h \geq 3$*, we use the terms "parabolic viscosity h-infinite supersolution", etc.*

In the case when $1 \leq h < 3$, the definition above is insufficient due to the singularity occurring when the horizontal gradient vanishes. Therefore, following [14] and [15], we define viscosity solutions to Equation (2.3) when $1 \leq h < 3$ as follows:

Definition 2. Let $\mathcal{O}_{t_1,t_2}$ be as above. A lower semicontinuous function $v : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$ is a *parabolic viscosity h-infinite supersolution* of $u_t - \Delta^h_\infty u = 0$ if whenever $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$ and $\phi \in \mathcal{B}u(p_0,t_0)$, we have

$$\begin{cases} \phi_t(p_0,t_0) - \Delta^h_\infty \phi(p_0,t_0) \geq 0 & \text{when } \nabla_0\phi(p_0,t_0) \neq 0 \\ \phi_t(p_0,t_0) - \min_{\|\eta\|=1} \langle (D^2\phi)^\star(p_0,t_0)\,\eta,\eta\rangle \geq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and } h=1 \\ \phi_t(p_0,t_0) \geq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and } 1<h<3 \end{cases}$$

An upper semicontinuous function $u : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$ is a *parabolic viscosity h-infinite subsolution* of $u_t - \Delta^h_\infty u = 0$ if whenever $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$ and $\phi \in \mathcal{A}u(p_0,t_0)$, we have

$$\begin{cases} \phi_t(p_0,t_0) - \Delta^h_\infty \phi(p_0,t_0) \leq 0 & \text{when } \nabla_0\phi(p_0,t_0) \neq 0 \\ \phi_t(p_0,t_0) - \max_{\|\eta\|=1} \langle (D^2\phi)^\star(p_0,t_0)\,\eta,\eta\rangle \leq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and } h=1 \\ \phi_t(p_0,t_0) \leq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and } 1<h<3 \end{cases}$$

A continuous function is a *parabolic viscosity h-infinite solution* if it is both a parabolic viscosity h-infinite subsolution and parabolic viscosity h-infinite subsolution.

Remark 3.4. *When* $1 < h < 3$*, we can actually consider the continuous operator*

$$(3.2)\, F^h_\infty(\nabla_0 u,(D^2u)^\star) = \begin{cases} -\|\nabla_0 u\|^{h-3}\langle (D^2u)^\star \nabla_0 u, \nabla_0 u\rangle = -\Delta^h_\infty u & \nabla_0 u \neq 0 \\ 0 & \nabla_0 u = 0. \end{cases}$$

Definitions 1 and 2 would then agree. (cf. [15]*)*

We also wish to define what [12] refers to as parabolic viscosity solutions. We first need to consider the set

$$\mathcal{A}^- u(p_0,t_0) = \{\phi \in \mathcal{C}^2(\mathcal{O}_{t_1,t_2}) : u(p,t) - \phi(p,t) \leq u(p_0,t_0) - \phi(p_0,t_0) = 0 \text{ for } p \neq p_0, t < t_0\}$$

consisting of all functions that touch from above only when $t < t_0$. Note that this set is larger than $\mathcal{A}u$ and corresponds physically to the past alone playing a role in determining the present. We define $\mathcal{B}^- u(p_0,t_0)$ similarly. We then have the following definition.

Definition 3. An upper semicontinuous function u on $\mathcal{O}_{t_1,t_2}$ is a *past parabolic viscosity subsolution* in $\mathcal{O}_{t_1,t_2}$ if $\phi \in \mathcal{A}^- u(p_0,t_0)$ produces

$$\phi_t(p_0,t_0) + F(t_0,p_0,u(p_0,t_0),\nabla_1\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star) \leq 0.$$

An lower semicontinuous function u on $\mathcal{O}_{t_1,t_2}$ is a *past parabolic viscosity supersolution* in $\mathcal{O}_{t_1,t_2}$ if $\phi \in \mathcal{B}^- u(p_0,t_0)$ produces

$$\phi_t(p_0,t_0) + F(t_0,p_0,u(p_0,t_0),\nabla_1\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star) \geq 0.$$

A continuous function is a *past parabolic viscosity solution* if it is both a past parabolic viscosity supersolution and subsolution.

We have the following proposition whose proof is obvious.

Proposition 3.5. *Past parabolic viscosity sub(super-)solutions are parabolic viscosity sub(super-)solutions. In particular, past parabolic viscosity h-infinite sub(super-)solutions are parabolic viscosity h-infinite subsub(super-)solutions for $h \geq 1$.*

3.4. **The Carnot Parabolic Maximum Principle.** In this subsection, we recall the Carnot Parabolic Maximum Principle and key corollaries, as proved in [6].

Lemma 3.6 (Carnot Parabolic Maximum Principle)**.** *Let u be a viscosity subsolution to Equation* (3.1) *and v be a viscosity supersolution to Equation* (3.1) *in the bounded parabolic set $\Omega \times (0,T)$ where Ω is a (bounded) domain and let τ be a positive real parameter. Let $\phi(p,q,t) = \varphi(p \cdot q^{-1},t)$ be a C^2 function in the space variables p and q and a C^1 function in t. Suppose the local maximum*

$$M_\tau \equiv \max_{\overline{\Omega}\times\overline{\Omega}\times[0,T]} \{u(p,t) - v(q,t) - \tau\phi(p,q,t)\} \tag{3.3}$$

occurs at the interior point (p_τ,q_τ,t_τ) of the parabolic set $\Omega\times\Omega\times(0,T)$. Define the $n\times n$ matrix W by

$$W_{ij} = X_i(p)X_j(q)\phi(p_\tau,q_\tau,t_\tau).$$

Let the $2n\times 2n$ matrix $\mathfrak{W}$ be given by

$$\mathfrak{W} = \begin{pmatrix} 0 & \frac{1}{2}(W - W^T) \\ \frac{1}{2}(W^T - W) & 0 \end{pmatrix} \tag{3.4}$$

and let the matrix $\mathcal{W} \in S^{2N}$ be given by

$$\mathcal{W} = \begin{pmatrix} D^2_{pp}\phi(p_\tau,q_\tau,t_\tau) & D^2_{pq}\phi(p_\tau,q_\tau,t_\tau) \\ \\ D^2_{qp}\phi(p_\tau,q_\tau,t_\tau) & D^2_{qq}\phi(p_\tau,q_\tau,t_\tau) \end{pmatrix}. \tag{3.5}$$

Suppose

$$\lim_{\tau\to\infty} \tau\phi(p_\tau,q_\tau,t_\tau) = 0.$$

Then for each $\tau > 0$, there exists real numbers a_1 and a_2, symmetric matrices $\mathcal{X}_\tau$ and $\mathcal{Y}_\tau$ and vector $\Upsilon_\tau \in V_1 \oplus V_2$, namely $\Upsilon_\tau = \nabla_1(p)\phi(p_\tau,q_\tau,t_\tau)$, so that the following hold:

A) $(a_1,\tau\Upsilon_\tau,\mathcal{X}_\tau) \in \overline{P}^{2,+}u(p_\tau,t_\tau)$ *and* $(a_2,\tau\Upsilon_\tau,\mathcal{Y}_\tau) \in \overline{P}^{2,-}v(q_\tau,t_\tau)$.
B) $a_1 - a_2 = \phi_t(p_\tau,q_\tau,t_\tau)$.

C) *For any vectors* $\xi, \epsilon \in V_1$, *we have*

$$\begin{aligned}\langle \mathcal{X}_\tau \xi, \xi\rangle - \langle \mathcal{Y}_\tau \epsilon, \epsilon\rangle \;&\leq\; \tau\langle (D_p^2\phi)^\star(p_\tau, q_\tau, t_\tau)(\xi - \epsilon), (\xi-\epsilon)\rangle + \tau\langle \mathfrak{W}(\xi\oplus\epsilon), (\xi\oplus\epsilon)\rangle \\ &\quad + \tau\|\mathcal{W}\|^2\|\mathbb{A}(\hat{p})^T\xi \oplus \mathbb{A}(\hat{q})^T\epsilon\|^2. \end{aligned} \tag{3.6}$$

In particular,

$$\langle \mathcal{X}_\tau \xi, \xi\rangle - \langle \mathcal{Y}_\tau \xi, \xi\rangle \lesssim \tau\|\mathcal{W}\|^2\|\xi\|^2. \tag{3.7}$$

Corollary 3.7. *Let* $\phi(p,q,t) = \phi(p,q) = \varphi(p\cdot q^{-1})$ *be independent of* t *and a non-negative function. Suppose* $\phi(p,q) = 0$ *exactly when* $p = q$. *Then*

$$\lim_{\tau\to\infty} \tau\phi(p_\tau, q_\tau) = 0.$$

In particular, if

$$\phi(p,q,t) = \frac{1}{m}\sum_{i=1}^{N}\left((p\cdot q^{-1})_i\right)^m \tag{3.8}$$

for some ***even*** *integer* $m \geq 4$ *where* $(p\cdot q^{-1})_i$ *is the* i*-th component of the Carnot group multiplication group law, then for the vector* Υ_τ *and matrices* $\mathcal{X}_\tau, \mathcal{Y}_\tau$, *from the Lemma, we have*

A) $(a_1, \tau\Upsilon_\tau, \mathcal{X}_\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau)$ *and* $(a_1, \tau\Upsilon_\tau, \mathcal{Y}_\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau)$.

B) *The vector* Υ_τ *satisfies*

$$\|\Upsilon_\tau\| \sim \phi(p_\tau, q_\tau)^{\frac{m-1}{m}}.$$

C) *For any fixed vector* $\xi \in V_1$, *we have*

$$\langle \mathcal{X}_\tau \xi, \xi\rangle - \langle \mathcal{Y}_\tau \xi, \xi\rangle \lesssim \tau\|\mathcal{W}\|^2\|\xi\|^2 \lesssim \tau(\phi(p_\tau, q_\tau))^{\frac{2m-4}{m}}\|\xi\|^2. \tag{3.9}$$

4. Uniqueness of viscosity solutions

We wish to formulate a comparison principle for the following problem.

Problem 4.1. *Let* $h \geq 1$. *Let* Ω *be a bounded domain and let* $\Omega_T = \Omega \times [0,T)$. *Let* $\psi \in C(\overline{\Omega})$ *and* $g \in C(\overline{\Omega_T})$. *We consider the following boundary and initial value problem:*

$$\begin{cases} u_t + F_\infty^h(\nabla_0 u, (D^2u)^\star) = 0 & \text{in } \ \Omega\times(0,T) & (E) \\ u(p,t) = g(p,t) & p\in\partial\Omega, \ t\in[0,T) & (BC) \\ u(p,0) = \psi(p) & p \in \overline{\Omega} & (IC) \end{cases} \tag{4.1}$$

We also adopt the definition that a subsolution $u(p,t)$ *to Problem 4.1 is a viscosity subsolution to (E),* $u(p,t) \leq g(p,t)$ *on* $\partial\Omega$ *with* $0 \leq t < T$ *and* $u(p,0) \leq \psi(p)$ *on* $\overline{\Omega}$. *Supersolutions and solutions are defined in an analogous matter.*

Because our solution u will be continuous, we offer the following remark:

Remark 4.2. *The functions* ψ *and* g *may be replaced by one function* $g \in C(\overline{\Omega_T})$. *This combines conditions (E) and (BC) into one condition*

$$u(p,t) = g(p,t), \quad (p,t) \in \partial_{\text{par}}\Omega_T \qquad (IBC) \tag{4.2}$$

Theorem 4.3. *Let Ω be a bounded domain in G and let $h \geq 1$. If u is a parabolic viscosity subsolution and v a parabolic viscosity supersolution to Problem* (4.1) *then $u \leq v$ on $\Omega_T \equiv \Omega \times [0,T)$.*

Proof. Our proof follows that of [8, Thm. 8.2] and so we discuss only the main parts.

For $\varepsilon > 0$, we substitute $\tilde{u} = u - \frac{\varepsilon}{T-t}$ for u and prove the theorem for

$$u_t + F_\infty^h(\nabla_0 u, (D^2u)^\star) \leq -\frac{\varepsilon}{T^2} < 0 \tag{4.3}$$

$$\lim_{t \uparrow T} u(p,t) = -\infty \quad \text{uniformly on} \quad \overline{\Omega} \tag{4.4}$$

and take limits to obtain the desired result. Assume the maximum occurs at $(p_0, t_0) \in \Omega \times (0,T)$ with

$$u(p_0,t_0) - v(p_0,t_0) = \delta > 0.$$

Case 1: $h > 1$.

Let $H \geq h+3$ be an even number. As in Equation (3.8), we let

$$\phi(p,q) = \frac{1}{H} \sum_{i=1}^N \big((p \cdot q^{-1})_i\big)^H$$

where $(p \cdot q^{-1})_i$ is the i-th component of the Carnot group multiplication group law. Let

$$M_\tau = u(p_\tau, t_\tau) - v(q_\tau, t_\tau) - \tau\phi(p_\tau, q_\tau)$$

with (p_τ, q_τ, t_τ) the maximum point in $\overline{\Omega} \times \overline{\Omega} \times [0,T)$ of $u(p,t) - v(q,t) - \tau\phi(p,q)$.

If $t_\tau = 0$, we have

$$0 < \delta \leq M_\tau \leq \sup_{\overline{\Omega} \times \overline{\Omega}} (\psi(p) - \psi(q) - \tau\phi(p,q))$$

leading to a contradiction for large τ. We therefore conclude $t_\tau > 0$ for large τ. Since $u \leq v$ on $\partial\Omega \times [0,T)$ by Equation (BC) of Problem (4.1), we conclude that for large τ, we have (p_τ, q_τ, t_τ) is an interior point. That is, $(p_\tau, q_\tau, t_\tau) \in \Omega \times \Omega \times (0,T)$. Using Corollary 3.7 Property A, we obtain

$$\begin{aligned} (a, \tau\Upsilon(p_\tau,q_\tau), \mathcal{X}_\tau) &\in \overline{P}^{2,+} u(p_\tau, t_\tau) \\ (a, \tau\Upsilon(p_\tau,q_\tau), \mathcal{Y}_\tau) &\in \overline{P}^{2,-} v(q_\tau, t_\tau) \end{aligned}$$

satisfying the equations

$$\begin{aligned} a + F_\infty^h(\tau\Upsilon(p_\tau,q_\tau), \mathcal{X}_\tau) &\leq -\frac{\varepsilon}{T^2} \\ a + F_\infty^h(\tau\Upsilon(p_\tau,q_\tau), \mathcal{Y}_\tau) &\geq 0. \end{aligned}$$

If there is a subsequence $\{p_\tau, q_\tau\}_{\tau>0}$ such that $p_\tau \neq q_\tau$, we subtract, and using Corollary 3.7, we have

$$\begin{aligned}
0 < \frac{\varepsilon}{T^2} &\leq (\tau\Upsilon(p_\tau,q_\tau))^{h-3}\tau^2\Big(\langle \mathcal{X}_\tau\Upsilon(p_\tau,q_\tau),\Upsilon(p_\tau,q_\tau)\rangle - \langle \mathcal{Y}_\tau\Upsilon(p_\tau,q_\tau),\Upsilon(p_\tau,q_\tau)\rangle\Big) \\
&\lesssim \tau^h\big(\varphi(p_\tau,q_\tau)^{\frac{H-1}{H}}\big)^{h-3}(\varphi(p_\tau,q_\tau))^{\frac{2H-4}{H}}(\varphi(p_\tau,q_\tau))^{\frac{2H-2}{H}} \qquad (4.5) \\
&= \tau^h(\varphi(p_\tau,q_\tau))^{\frac{Hh+H-h-3}{H}} = (\tau\varphi(p_\tau,q_\tau))^h\varphi(p_\tau,q_\tau)^{\frac{H-h-3}{H}}. \qquad (4.6)
\end{aligned}$$

Because $H > h+3$, we arrive at a contradiction as $\tau \to \infty$.

If we have $p_\tau = q_\tau$, we arrive at a contradiction since

$$F^h_\infty(\tau\Upsilon(p_\tau,q_\tau),\mathcal{X}_\tau) = F^h_\infty(\tau\Upsilon(p_\tau,q_\tau),\mathcal{Y}_\tau) = 0.$$

Case 2: $h = 1$.

We follow the proof of Theorem 3.1 in [14]. We let

$$\varphi(p,q,t,s) = \frac{1}{4}\sum_{i=1}^N \big((p\cdot q^{-1})_i\big)^4 + \frac{1}{2}(t-s)^2$$

and let $(p_\tau, q_\tau, t_\tau, s_\tau)$ be the maximum of

$$u(p,t) - v(q,s) - \tau\phi(p,q,t,s)$$

Again, for large τ, this point is an interior point. If we have a sequence where $p_\tau \neq q_\tau$, then Lemma 3.2 yields

$$\begin{aligned}
(\tau(t_\tau - s_\tau), \tau\Upsilon(p_\tau,q_\tau), \mathcal{X}_\tau) &\in \overline{P}^{2,+}u(p_\tau,t_\tau) \\
(\tau(t_\tau - s_\tau), \tau\Upsilon(p_\tau,q_\tau), \mathcal{Y}_\tau) &\in \overline{P}^{2,-}v(q_\tau,s_\tau)
\end{aligned}$$

satisfying the equations

$$\begin{aligned}
\tau(t_\tau - s_\tau) + F^h_\infty(\tau\Upsilon(p_\tau,q_\tau),\mathcal{X}_\tau) &\leq -\frac{\varepsilon}{T^2} \\
\tau(t_\tau - s_\tau) + F^h_\infty(\tau\Upsilon(p_\tau,q_\tau),\mathcal{Y}_\tau) &\geq 0.
\end{aligned}$$

As in the first case, we subtract to obtain

$$\begin{aligned}
0 < \frac{\varepsilon}{T^2} &\leq (\tau\Upsilon(p_\tau,q_\tau))^{-2}\tau^2\Big(\langle \mathcal{X}_\tau\Upsilon(p_\tau,q_\tau),\Upsilon(p_\tau,q_\tau)\rangle - \langle \mathcal{Y}_\tau\Upsilon(p_\tau,q_\tau),\Upsilon(p_\tau,q_\tau)\rangle\Big) \\
&\lesssim \varphi(p_\tau,q_\tau)^{-\frac{3}{2}}(\tau\varphi(p_\tau,q_\tau)\varphi(p_\tau,q_\tau)^{\frac{3}{2}}) = \tau\varphi(p_\tau,q_\tau).
\end{aligned}$$

We arrive at a contradiction as $\tau \to \infty$.

If $p_\tau = q_\tau$, then $v(q,s) - \beta^v(q,s)$ has a local minimum at (q_τ, s_τ) where

$$\beta^v(q,s) = -\frac{\tau}{4}\sum_{i=1}^N \big((p_\tau\cdot q^{-1})_i\big)^4 - \frac{\tau}{2}(t_\tau - s)^2.$$

We then have

$$0 < \varepsilon(T-s_\tau)^{-2} \leq \beta^v_s(q_\tau,s_\tau) - \min_{\|\eta\|=1}\langle (D^2\beta^v)^\star(q_\tau,s_\tau)\,\eta,\eta\rangle.$$

Similarly, $u(p,t) - \beta^u(p,t)$ has a local maximum at (p_τ, t_τ) where

$$\beta^u(p,t) = \frac{\tau}{4}\sum_{i=1}^{N}\big((p\cdot q_\tau^{-1})_i\big)^4 + \frac{\tau}{2}(t-s_\tau)^2.$$

We then have

$$0 \geq \beta^u_t(p_\tau,t_\tau) - \max_{\|\eta\|=1}\langle (D^2\beta^u)^\star(p_\tau,t_\tau)\,\eta,\eta\rangle$$

and subtraction gives us

$$\begin{aligned}
0 < \varepsilon(T-s_\tau)^{-2} \quad &\leq \quad \max_{\|\eta\|=1}\langle (D^2\beta^u)^\star(p_\tau,t_\tau)\,\eta,\eta\rangle - \min_{\|\eta\|=1}\langle (D^2\beta^v)^\star(q_\tau,s_\tau)\,\eta,\eta\rangle \\
&\quad\ + \beta^v_s(q_\tau,s_\tau) - \beta^u_t(p_\tau,t_\tau) \\
&= \quad \tau\max_{\|\eta\|=1}\langle (D^2_{pp}\varphi(p\cdot q_\tau^{-1}))^\star(p_\tau,t_\tau)\,\eta,\eta\rangle \\
&\quad\ - \tau\min_{\|\eta\|=1}\langle (D^2_{qq}\varphi(p_\tau\cdot q^{-1}))^\star(q_\tau,s_\tau)\,\eta,\eta\rangle \\
&\quad\ + \tau(t_\tau-s_\tau) - \tau(t_\tau-s_\tau) \\
&= \quad 0.
\end{aligned}$$

Here, the last equality comes from the fact that $p_\tau = q_\tau$ and the definition of $\varphi(p\cdot q^{-1})$. ☐

The comparison principle has the following consequences concerning properties of solutions:

Corollary 4.4. *Let $h \geq 1$. The past parabolic viscosity h-infinite solutions are exactly the parabolic viscosity h-infinite solutions.*

Proof. By Proposition 3.5, past parabolic viscosity h-infinite sub(super-)solutions are parabolic viscosity h-infinite sub(super-)solutions. To prove the converse, we will follow the proof of the subsolution case found in [12], highlighting the main details. Assume that u is not a past parabolic viscosity h-infinite subsolution. Let $\phi \in \mathcal{A}^-u(p_0,t_0)$ have the property that

$$\phi_t(p_0,t_0) - \Delta^h_\infty\phi(p_0,t_0) \geq \epsilon > 0$$

for a small parameter ϵ. We may assume p_0 is the origin. Let $r>0$ and define $S_r = B_{\mathcal{N}}(r)\times(t_0-r,t_0)$ and let ∂S_r be its parabolic boundary. Then the function

$$\tilde{\phi}_r(p,t) = \phi(p,t) + (t_0-t)^{8l!} - r^{8l!} + (\mathcal{N}(p))^{8l!}$$

is a classical supersolution for sufficiently small r. We then observe that $u \leq \tilde{\phi}_r$ on ∂S_r but $u(0,t_0) > \tilde{\phi}(0,t_0)$. Thus, the comparison prinicple, Theorem 4.3, does not hold. Thus, u is not a parabolic viscosity h-infinite subsolution. The supersolution case is identical and omitted. ☐

The following corollary has a proof similar to [14, Lemma 3.2].

Corollary 4.5. *Let $u : \Omega_T \to \mathbb{R}$ be upper semicontinuous. Let $\phi \in \mathcal{A}u(p_0, t_0)$. If*

$$(4.7) \begin{cases} \phi_t(p_0,t_0) - \Delta^1_\infty \phi(p_0,t_0) \leq 0 & \textit{when } \ \nabla_0\phi(p_0,t_0) \neq 0 \\ \phi_t(p_0,t_0) \leq 0 & \textit{when } \ \nabla_0\phi(p_0,t_0) = 0, (D^2\phi)^\star(p_0,t_0) = 0 \end{cases}$$

then u is a viscosity subsolution to (E) of Problem (4.1)*.*

We also have the following function estimates with respect to boundary data.

Corollary 4.6. *Let $h \geq 1$. Let $g_1, g_2 \in C(\overline{\Omega_T})$ and u_1, u_2 be parabolic viscosity solutions to Equation 4.1 with boundary data g_1 and g_2, respectively. Then*

$$\sup_{(p,t)\in\Omega_T} |u_1(p,t) - u_2(p,t)| \leq \sup_{(p,t)\in\partial_{\text{par}}\Omega_T} |g_1(p,t) - g_2(p,t)|.$$

Proof. The function $u^+(p,t) = u_2(p,t) + \sup_{(p,t)\in\partial_{\text{par}}\Omega_T} |g_1(p,t) - g_2(p,t)|$ is a parabolic viscosity supersolution with boundary data g_1 and the function $u^-(p,t) = u_2(p,t) - \sup_{(p,t)\in\partial_{\text{par}}\Omega_T} |g_1(p,t) - g_2(p,t)|$ is a parabolic viscosity subsolution with boundary data g_1. Moreover, $u^- \leq u_1 \leq u^+$ on $\partial_{\text{par}}\Omega_T$ and by Theorem 4.3 $u^- \leq u_1 \leq u^+$ in Ω_T. □

Corollary 4.7. *Let $h \geq 1$. Let $g \in C(\overline{\Omega_T})$. Then every parabolic viscosity solution to Problem 4.1 satisfies*

$$\sup_{(p,t)\in\Omega_T} |u(p,t)| \leq \sup_{(p,t)\in\partial_{\text{par}}\Omega_T} |g(p,t)|$$

Proof. The proof is similar to the previous corollary, but using the functions $u^\pm(p,t) = \pm\sup_{(p,t)\in\partial_{\text{par}}\Omega_T} |g(p,t)|$ instead. □

5. Existence of Viscosity Solutions

5.1. **Parabolic Viscosity Infinite Solutions: The Continuity Case.** As above, we will focus on the equations of the form (3.1) for continuous and proper $F : [0,T]\times G\times\mathbb{R}\times g\times S^{n_1} \to \mathbb{R}$ that possess a comparison principle such as Theorem 4.3 or [6, Thm. 3.6]. We will use Perron's method combined with the Carnot Parabolic Maximum Principle to yield the desired existence theorem. In particular, the following proofs are similar to those found in [10, Chapter 2] except that the Euclidean derivatives have been replaced with horizontal derivatives and the Euclidean norms have been replaced with the gauge norm.

Lemma 5.1. *Let $\mathcal{L}$ be a collection of parabolic viscosity supersolutions to* (3.1) *and let $u(p,t) = \inf\{v(p,t) : v \in \mathcal{L}\}$. If u is finite in a dense subset of $\Omega_T = \Omega \times [0,T)$ then u is a parabolic viscosity supersolution to* (3.1)*.*

Proof. First note that u is lower semicontinous since every $v \in \mathcal{L}$ is. Let $(p_0,t_0) \in \Omega_T$ and $\phi \in \mathcal{A}u(p_0,t_0)$. Now let

$$\psi(p,t) = \phi(p,t) - (d_{\mathcal{N}}(p_0,p))^{2l!} - |t-t_0|^2$$

and notice that $\psi \in \mathcal{A}u(p_0,t_0)$. Then

$$\begin{aligned}(u-\psi)(p,t)-(d_{\mathcal{N}}(p_0,p))^{2l!}-|t-t_0|^2 &= (u-\phi)(p,t)\\ &\geq (u-\phi)(p_0,t_0)\\ &= (u-\psi)(p_0,t_0)\\ &= 0\end{aligned}$$

yields

$$(u-\psi)(p,t)\geq (d_{\mathcal{N}}(p_0,p))^{2l!}+|t-t_0|^2. \tag{5.1}$$

Since u is lower semicontinuous, there exists a sequence $\{(p_k,t_k)\}$ with $t_k<t_0$ converging to (p_0,t_0) as $k\to\infty$ such that

$$(u-\psi)(p_k,t_k)\to(u-\psi)(p_0,t_0)=0.$$

Since $u(p,t)=\inf\{v(p,t): v\in\mathcal{L}\}$, there exists a sequence $\{v_k\}\subset\mathcal{L}$ such that $v_k(p_k,t_k)<u(p_k,t_k)+1/k$ for k = 1,2, $\ldots$. Since $v_k\geq u$, (5.1) gives us

$$(v_k-\psi)(p,t)\geq(u-\psi)(p,t)\geq (d_{\mathcal{N}}(p_0,p))^{2l!}+|t-t_0|^2. \tag{5.2}$$

Let $B\subset\Omega$ denote a compact neighborhood of (p_0,t_0). Since $v_k-\psi$ is lower semicontinuous, it attains a minimum in B at a point $(q_k,s_k)\in B$. Then by (5.1) and (5.2) we have

$$(u-\psi)(p_k,t_k)+1/k>(v_k-\psi)(p_k,t_k)\geq(v_k-\psi)(q_k,s_k)\geq(d_{\mathcal{N}}(p_0,q_k))^{2l!}+|s_k-t_0|^2\geq 0$$

for sufficiently large k such that $(p_k,t_k)\in B$. By the squeeze theorem, $(q_k,s_k)\to(p_0,t_0)$ as $k\to\infty$. Let $\eta=\psi-(d_{\mathcal{N}}(q_k,p))^{2l!}-|s_k-t|^2$. Then $\eta\in\mathcal{A}v_k(q_k,s_k)$ and we have that

$$\eta_t(q_k,s_k)+F(s_k,q_k,v_k(q_k,s_k),\nabla_1\eta(q_k,s_k),(D^2\eta(q_k,s_k))^\star)\geq 0.$$

This implies

$$\psi_t(q_k,s_k)+F(s_k,q_k,v_k(q_k,s_k),\nabla_1\psi(q_k,s_k),(D^2\psi(s_k,s_k))^\star)\geq 0.$$

Letting $k\to\infty$ yields

$$\phi_t(p_0,t_0)+F(t_0,p_0,u(p_0,t_0)\nabla_1\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star)\geq 0.$$

and that u is a parabolic viscosity supersolution as desired. □

A similar argument yields the following.

Lemma 5.2. *Let $\mathcal{L}$ be a collection of parabolic viscosity subsolutions to* (3.1) *and let $u(p,t)=\sup\{v(p,t): v\in\mathcal{L}\}$. If u is finite in a dense subset of Ω_T then u is a parabolic viscosity subsolution to* (3.1).

For the following lemmas, we need to recall the following definition.

Definition 4. The *upper and lower semi-continuous envelopes* of a function u are given by

$$u^*(p,t) := \lim_{r\downarrow 0} \sup\{u(q,s) : |q^{-1}p|_g + |s-t| \le r\}$$

and

$$u_*(p,t) := \lim_{r\downarrow 0} \inf\{u(q,s) : |q^{-1}p|_g + |s-t| \le r\},$$

respectively.

Lemma 5.3. *Let h be a parabolic viscosity supersolution to* (3.1) *in Ω_T. Let $\mathcal{S}$ be the collection of all parabolic viscosity subsolutions v of* (3.1) *satisfying $v \le h$. If for $\hat{v} \in \mathcal{S}$, $\hat{v}_*$ is not a parabolic viscosity supersolution of* (3.1) *then there is a function $w \in \mathcal{S}$ and a point (p_0, t_0) such that $\hat{v}(p_0,t_0) < w(p_0,t_0)$.*

Proof. Let $\hat{v} \in \mathcal{S}$ such that $\hat{v}_*$ is not a parabolic viscosity supersolution of (3.1). Then there exists $(\hat{p},\hat{t}) \in \Omega_T$ and $\phi \in \mathcal{A}\hat{v}_*(\hat{p},\hat{t})$ such that

$$\phi_t(p,t) + F(t,p,\hat{v}_*(p,t), \nabla_1\phi(p,t), (D^2\phi(p,t))^\star) > 0. \tag{5.3}$$

Let

$$\psi(p,t) = \phi(p,t) - (d_{\mathcal{N}}(\hat{p},p))^{2l!} - |t-\hat{t}|^2$$

and notice that $\psi \in \mathcal{A}\hat{v}_*(\hat{p},\hat{t})$. As in Lemma 5.1,

$$(\hat{v}_* - \psi)(p,t) \ge (d_{\mathcal{N}}(\hat{p},p))^{2l!} + |t-\hat{t}|^2. \tag{5.4}$$

Let B denote a compact neighborhood of $(\hat{p},\hat{t})$ and let

$$B_{k\epsilon} = B \cap \left\{(p,t) : (d_{\mathcal{N}}(\hat{p},p))^{2l!} \le k\epsilon \text{ and } |t-\hat{t}|^2 \le k\epsilon\right\}.$$

Since $\hat{v} \in \mathcal{S}$, we have that $\hat{v} \le h$ and thus $\psi(\hat{p},\hat{t}) = \hat{v}_*(\hat{p},\hat{t}) \le \hat{v}(\hat{p},\hat{t}) \le h(\hat{p},\hat{t})$. However, if $\psi(\hat{p},\hat{t}) = h(\hat{p},\hat{t})$, then $\psi \in \mathcal{A}h(\hat{p},\hat{t})$ and inequality (5.3) would be contradictory. Thus,

$$\psi(\hat{p},\hat{t}) < h(\hat{p},\hat{t}).$$

Since ψ is continuous and h is lower semicontinuous, there exists $\epsilon > 0$ such that

$$\psi(p,t) + 4\epsilon \le h(p,t)$$

for $(p,t) \in B_{2\epsilon}$. Notice that $\psi + 4\epsilon$ is a subsolution of (3.1) on the interior of $B_{2\epsilon}$. Further, by (5.4)

$$\hat{v}(p,t) \ge \hat{v}_*(p,t) \ge \psi(p,t) + 4\epsilon \text{ for } (p,t) \in B_{2\epsilon}\backslash B_\epsilon. \tag{5.5}$$

We now define ω by

$$\omega = \begin{cases} \max\{\psi(p,t)+4\epsilon, \hat{v}(p,t)\} & (p,t) \in B_\epsilon \\ \hat{v}(p,t) & (p,t) \in \Omega_T\backslash B_\epsilon \end{cases}$$

But by (5.5)

$$\omega(p,t) = \max\{\psi(p,t)+4\epsilon, \hat{v}(p,t)\} \text{ for } (p,t) \in B_{2\epsilon},$$

not just for $(p,t) \in B_\epsilon$. Then by Lemma 5.2, ω is a subsolution in the interior of $B_{2\epsilon}$ and thus a subsolution in Ω_T. Therefore, $\omega \in \mathcal{S}$. Since

$$0 = (\hat{v}_* - \psi)(\hat{p}, \hat{t}) = \lim_{r \downarrow 0} \inf \{(\hat{v} - \psi)(p,t) : (p,t) \in B_r\}$$

there is a point $(p_0, t_0) \in B_\epsilon$ that satisfies

$$\hat{v}(p_0,t_0) - \psi(p_0,t_0) < 4\epsilon$$

which yields

$$\hat{v}(p_0,t_0) < \psi(p_0,t_0) + 4\epsilon = \omega(p_0,t_0).$$

Thus, we have constructed $\omega \in \mathcal{S}$ that satisfies $\hat{v}(p_0,t_0) < \omega(p_0,t_0)$. $\square$

We then have the following existence theorem concerning parabolic viscosity solutions.

Theorem 5.4. *Let f be a parabolic viscosity subsolution to* (3.1) *and g be a parabolic viscosity supersolution to* (3.1) *satisfying $f \le g$ on Ω_T and $f_* = g^*$ on $\partial_{\text{par}}\mathcal{O}_{0,T}$. Then there is a parabolic viscosity solution u to* (3.1) *satisfying $u \in C(\overline{O_T})$. Explicitly, there exists a unique parabolic viscosity infinite solution to Problem 4.1 when $h > 1$.*

Proof. Let

$$S = \{\nu : \ \nu \text{ is a parabolic viscosity subsolution to (3.1) in } \Omega_T \text{ with } \nu \le g \text{ in } \Omega_T\}$$

and

$$u(p,t) = \sup\{\nu(p,t) : \ \nu \in S\}.$$

Since $f \le g$, the set S is nonempty. Notice that $f \le u \le g$ by construction. By Lemma (5.2), u is a parabolic viscosity subsolution. Suppose u_* is not a parabolic viscosity supersolution. Then by Lemma 5.3, there exists a function $w \in S$ and a point $(p_0,t_0) \in \Omega_T$ such that $u(p_0,t_0) < w(p_0,t_0)$. But this contradicts the definition of u at (p_0,t_0). Thus u_* is a parabolic viscosity supersolution. By our assumptions on f and g on $\partial_{\text{par}}\mathcal{O}_{0,T}$,

$$u = u^* \le g^* = f_* \le u_*$$

on $\partial_{\text{par}}\mathcal{O}_{0,T}$. Then by the (assumed) comparison principle, $u \le u_*$ on Ω_T. Thus we have u is a parabolic viscosity solution such that $u \in C(\overline{O_T})$. $\square$

5.2. **The $h = 1$ case.** We begin by recalling the definition of upper and lower relaxed limit of a function. [8, 10].

Definition 5. For $\varepsilon > 0$, consider the function $h_\varepsilon : O_T \subset G \to \mathbb{R}$. The *upper relaxed limit* $\overline{h}(p,t)$ and the lower relaxed limit $\underline{h}(p,t)$ are given by

$$\begin{aligned}
\overline{h}(p,t) &= \limsup_{\hat{p}\to p,\hat{t}\to t,\varepsilon\to 0} h_\varepsilon(\hat{p},\hat{t}) \\
&= \lim_{\varepsilon\to 0} \sup_{0<\delta<\varepsilon} \{h_\delta(\hat{p},\hat{t}) : O_T \cap B_\varepsilon(\hat{p},\hat{t})\} \\
\text{and } \underline{h}(p,t) &= \liminf_{\hat{p}\to p,\hat{t}\to t,\varepsilon\to 0} h_\varepsilon(\hat{p},\hat{t}) \\
&= \lim_{\varepsilon\to 0} \inf_{0<\delta<\varepsilon} \{h_\delta(\hat{p},\hat{t}) : O_T \cap B_\varepsilon(\hat{p},\hat{t})\}
\end{aligned}$$

Taking the relaxed limits as $h \to 1^+$ of the operator $F^h_\infty(\nabla_0 u, (D^2u)^\star)$ in Equation 3.2, we have via the continuity of the operator

$$\overline{F}^1_\infty(\nabla_0 u, (D^2u)^\star) = \underline{F}^1_\infty(\nabla_0 u, (D^2u)^\star) = \begin{cases} -\|\nabla_0 u\|^{-2}\langle (D^2u)^\star \nabla_0 u, \nabla_0 u\rangle & \nabla_0 u \neq 0 \\ 0 & \nabla_0 u = 0. \end{cases}$$

We give this operator the label $\mathcal{F}(\nabla_0 u, (D^2u)^\star)$. Consider the relaxed limits $\overline{u}(p,t)$ and $\underline{u}(p,t)$ of the sequence of unique (continuous) viscosity solutions to Problem 4.1 $\{u_h(p,t)\}$ as $h \to 1^+$. By [10, Thm 2.2.1], we have $\overline{u}(p,t)$ is a viscosity subsolution and $\underline{u}(p,t)$ is a viscosity supersolution to

$$u_t + \mathcal{F}(\nabla_0 u, (D^2u)^\star) = 0.$$

We have the following comparison principle, whose proof is similar to Theorem 4.3 in the case to $h = 1$ and is omitted.

Lemma 5.5. *Let Ω be a bounded domain in G. If $\mathfrak{u}$ is a parabolic viscosity subsolution and $\mathfrak{v}$ a parabolic viscosity supersolution to*

$$u_t + \mathcal{F}(\nabla_0 u, (D^2u)^\star) = 0.$$

then $\mathfrak{u} \leq \mathfrak{v}$ on $\Omega_T \equiv \Omega \times [0,T)$.

Corollary 5.6. *$\overline{u}(p,t) = \underline{u}(p,t)$.*

Proof. By construction, $\underline{u}(p,t) \leq \overline{u}(p,t)$. By the Lemma, $\underline{u}(p,t) \geq \overline{u}(p,t)$. □

Using the corollary, we will call this common relaxed limit $u^1(p,t)$. By [10, Chapter 2] and [8, Section 6], it is continuous and the sequence $\{u_h(p,t)\}$ converges locally uniformly to $u^1(p,t)$ as $h \to 1^+$.

We then have the following theorem.

Theorem 5.7. *There exists a unique parabolic viscosity infinite solution to Problem 4.1 when $h = 1$.*

Proof. Let $\{u_h(p,t)\}$ and $u^1(p,t)$ be as above. Let $\{h_j\}$ be a subsequence with $h_j \to 1^+$ where $u_h(p,t) \to u^1(p,t)$ uniformly. We may assume $h_j < 3$.

Let $\phi \in \mathcal{A}u_1(p_0,t_0)$. Using the uniform convergence, there is a sequence $\{p_j,t_j\} \to (p_0,t_0)$ so that $\phi \in \mathcal{A}u_{h_j}(p_j,t_j)$. If $\nabla_0\phi(p_0,t_0) \neq 0$, we have $\nabla_0\phi(p_j,t_j) \neq 0$ for sufficiently large j. We then have

$$\phi_t(p_j,t_j) - \Delta_\infty^{h_j}\phi(p_j,t_j) \leq 0$$

and letting $j \to \infty$ yields

$$\phi_t(p_0,t_0) - \Delta_\infty^{1}\phi(p_0,t_0) \leq 0.$$

Suppose $\nabla_0\phi(p_0,t_0) = 0$. By Corollary 4.5, we may assume $(D^2\phi)^\star(p_0,t_0) = 0$. Suppose passing to a subsequence if needed, we have $\nabla_0\phi(p_j,t_j) \neq 0$. Then

$$\phi_t(p_j,t_j) - \max_{\|\eta\|=1}\langle (D^2\phi)^\star(p_j,t_j)\ \eta,\eta\rangle \leq \phi_t(p_j,t_j) - \Delta_\infty^{h_j}\phi(p_j,t_j) \leq 0.$$

Letting $j \to \infty$ yields

$$\phi_t(p_0,t_0) = \phi_t(p_j,t_j) - \max_{\|\eta\|=1}\langle (D^2\phi)^\star(p_0,t_0)\ \eta,\eta\rangle \leq 0.$$

In the case $\nabla_0\phi(p_j,t_j) = 0$, since $h_j < 3$, we have $\phi_t(p_j,t_j) \leq 0$ and letting $j \to \infty$ yields $\phi_t(p_0,t_0) \leq 0$. We conclude that u_1 is a parabolic viscosity h-infinite subsolution. Similarly, u_1 is a parabolic viscosity h-infinite supersolution. □

6. The limit as $t \to \infty$.

We now focus our attention on the asymptotic limits of the parabolic viscosity h-infinite solutions. We wish to show that for $1 \leq h$, we have the (unique) viscosity solution to $u_t - \Delta_\infty^h u = 0$ approaches the viscosity solution of $-\Delta_\infty^h u = 0$ as $t \to \infty$. Our goal is the following theorem:

Theorem 6.1. *Let $h > 1$ and $u \in C(\overline{\Omega} \times [0,\infty))$ be a viscosity solution of*

$$\begin{cases} u_t - \Delta_\infty^h u = 0 & \text{in } \Omega \times (0,\infty), \\ u(p,t) = g(p) & \text{on } \partial_{\text{par}}(\Omega \times (0,\infty)) \end{cases} \tag{6.1}$$

with $g : \overline{\Omega} \to \mathbb{R}$ continuous and assuming that $\partial\Omega$ satisfies the property of positive geometric density (see [12, pg. 2909]*). Then $u(p,t) \to U(p)$ uniformly in Ω as $t \to \infty$ where $U(p)$ is the unique viscosity solution of $-\Delta_\infty^h U = 0$ with the Dirichlet boundary condition $lim_{q\to p}U(q) = g(p)$ for all $p \in \partial\Omega$.*

We first must establish the uniqueness of viscosity solutions to the limit equation. Note that for future reference, we include the case $h = 1$.

Theorem 6.2. *Let $1 \leq h < \infty$ and let Ω be a bounded domain. Let u be a viscosity subsolution to $\Delta_\infty^h u = 0$ and let v be a viscosity supersolution to $-\Delta_\infty^h u = 0$. Then,*

$$\sup_{p\in\overline{\Omega}}(u(p) - v(p)) = \sup_{p\in\partial\Omega}(u(p) - v(p)).$$

Proof. Let u be a viscosity subsolution to $-\Delta_\infty^h u = 0$. Then choose $\phi \in \mathcal{C}^2_{\text{sub}}(\Omega)$ such that $0 = \phi(p_0) - u(p_0) < \phi(p) - u(p)$ for $p \in \Omega$, $p \neq p_0$. If $\|\nabla_0\phi(p_0)\| = 0$, then $-\langle (D^2\phi)^\star(p_0)\nabla_0\phi(p_0), \nabla_0\phi(p_0)\rangle = 0 \leq 0$. If $\|\nabla_0\phi(p_0)\| \neq 0$, we then have

$$-\Delta_\infty^h \phi(p_0) = -\|\nabla_0\phi(p_0)\|^{h-3}\langle (D^2\phi)^\star(p_0)\nabla_0\phi(p_0), \nabla_0\phi(p_0)\rangle \leq 0.$$

Dividing, we have $-\langle (D^2\phi)^\star(p_0)\nabla_0\phi(p_0), \nabla_0\phi(p_0)\rangle \leq 0$. In either case, u is a viscosity subsolution to $-\Delta_\infty^3 u = 0$. Similarly, v is a viscosity supersolution to $-\Delta_\infty^3 u = 0$. The theorem follows from the corresponding result for $-\Delta_\infty^3 u = 0$ in [5, 3, 16]. □

We state some obvious corollaries:

Corollary 6.3. *Let $1 \leq h < \infty$ and let $g : \partial\Omega \to \mathbb{R}$ be continuous. Then there exactly one solution to*

$$\begin{cases} -\Delta_\infty^h u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Corollary 6.4. *Let $1 \leq h < \infty$ and let $g : \partial\Omega \to \mathbb{R}$ be continuous. The unique viscosity solution to*

$$\begin{cases} -\Delta_\infty^h u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is the unique viscosity solution to

$$\begin{cases} -\Delta_\infty^3 u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Our method of proof for Theorem 6.1 follows that of [12, Theorem 2], the core of which hinges on the construction of a parabolic test function from an elliptic one. In order to construct such a parabolic test function, we need to examine the homogeneity of Equation (6.1). A quick calculation shows that for a fixed $h > 1$, $k^{\frac{1}{h-1}}u(x, kt)$ is a $\mathcal{C}^2_{\text{sub}}$ solution to Equation (6.1) if $u(x,t)$ is a $\mathcal{C}^2_{\text{sub}}$ solution. A routine calculation then shows parabolic viscosity h-infinite solutions share this homogeneity. We use this property in the following lemma, the proof of which can be found in [9, pg. 170]. (Also, cf. [6, Lemma 6.2] and [12].)

Lemma 6.5. *Let u be as in Theorem 6.1 and $h > 1$. Then for every $(x,t) \in \Omega \times (0,\infty)$ and for $0 < \mathcal{T} < t$, we have*

$$|u(x, t - \mathcal{T}) - u(x,t)| \leq \frac{2||g||_{\infty,\Omega}}{h-1}\left(1 - \frac{\mathcal{T}}{t}\right)^{\frac{h}{1-h}}\frac{\mathcal{T}}{t}$$

Proof. [**Theorem 6.1**] Fix $h > 1$. Let u be a viscosity solution of (6.1). The results of [9, Chapter III] imply that the family $\{u(\cdot, t) : t \in (0,\infty)\}$ is equicontinuous. Since it is uniformly bounded due to the boundedness of g, Arzela-Ascoli's theorem yields that there exists a sequence $t_j \to \infty$ such that $u(\cdot, t_j)$ converge uniformly in $\overline{\Omega}$ to a function $U \in C(\overline{\Omega})$ for which $U(p) = g(p)$ for all $p \in \partial\Omega$. Since it is known from [5, Lemma 5.5] that the Dirichlet problem for the subelliptic $\mathtt{p}$-Laplace equation possesses a unique solution, it is enough to show that U is a viscosity $\mathtt{p}$-subsolution to $-\Delta_{\mathtt{p}} U = 0$ on Ω. With

that in mind, let $p_0 \in \Omega$ and choose $\phi \in \mathcal{C}^2_{\text{sub}}(\Omega)$ such that $0 = \phi(p_0) - U(p_0) < \phi(p) - U(p)$ for $p \in \Omega$, $p \neq p_0$. Using the uniform convergence, we can find a sequence $p_j \to p_0$ such that $u(\cdot, t_j) - \phi$ has a local maximum at p_j. Now define

$$\phi_j(p,t) = \phi(p) + C\left(\frac{t}{t_j}\right)^{\frac{h}{1-h}} \frac{t_j - t}{t_j},$$

where $C = 2||g||_{\infty,\Omega}/(h-1)$. Note that $\phi_j(p,t) \in \mathcal{C}^2_{\text{sub}}(\Omega \times (0,\infty))$. Then using Lemma 6.5,

$$\begin{aligned} u(p_j,t_j) - \phi_j(p_j,t_j) &= u(p_j,t_j) - \phi(p_j) \geq u(p,t_j) - \phi(p) \\ &\geq u(p,t) - \phi(p) - C\left(\frac{t}{t_j}\right)^{\frac{h}{1-h}} \frac{t_j - t}{t_j} \\ &= u(p,t) - \phi_j(p,t) \end{aligned}$$

for any $p \in \Omega$ and $0 < t < t_j$. Thus we have that ϕ_j is an admissible test function at (p_j, t_j) on $\Omega \times [0,T]$. Therefore,

$$(\phi_j)_t(p_j,t_j) - \Delta^h_\infty \phi_j(p_j,t_j) \leq 0.$$

This yields

$$-\Delta^h_\infty \phi(p_j) \leq \frac{C}{t_j}.$$

The theorem follows by letting $j \to \infty$. □

Combining the results of the previous sections, we have the following theorem:

Theorem 6.6. *The following diagram commutes:*

$$\begin{array}{ccc} u^{h,t}_t - \Delta^h_\infty u^{h,t} = 0 & \xrightarrow[h\to 1^+]{} & u^{1,t}_t - \Delta^1_\infty u^{1,t} = 0 \\ \Big\downarrow {\scriptstyle t\to\infty} & & \Big\downarrow {\scriptstyle t\to\infty} \\ -\Delta^h_\infty u^{h,\infty} = 0 & \xrightarrow[h\to 1^+]{} & -\Delta^1_\infty u^{1,\infty} = 0 \end{array}$$

Proof. By Theorem 6.1, Corollary 6.4, and Theorem 5.7, the top, bottom and left limits exist, with the left limit being a uniform limit. By results of iterated limits (see, for example, [1]), we have the fourth limit exists, as does the full limit. In particular,

$$\lim_{\substack{h\to 1^+ \\ t\to\infty}} u^{h,t} = \lim_{h\to 1^+} \lim_{t\to\infty} u^{h,t} = \lim_{t\to\infty} \lim_{h\to 1^+} u^{h,t} = u^{1,\infty}$$

□

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